JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 18, No. 3, May–June 1995

Nonlinear Modal Control Method

Joseph C. Slater*
Wright State University, Dayton, Ohio 45435
and
Daniel J. Inman†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

This work extends previous work on nonlinear normal modes to include the case of forced response. This allows the nonlinear normal mode method to be applied to the feedback control problem providing a new method of controlling nonlinear multiple-degree-of-freedom systems. The proposed method uses a transformation proposed earlier for homogeneous systems written in state-space form. The coordinate transformation for the forcing vector is defined here in state space and related to the physical coordinate system. The result is a pseudomodal decoupling transformation of a nonlinear inhomogeneous system. Although interesting in its own statement, this transformation also provides a nonlinear modal control scheme. This result is applied to a known coupled two degree-of-freedom oscillator with a cubic stiffness term. The results illustrate the design of a nonlinear modal control law.

I. Introduction

THE concept that nonlinear modes with nonlinear modal equations exist for some set of nonlinear systems has been accepted intuitively by many for quite some time. It was not until 1964 when Rosenberg presented the first paper on nonsimilar normal modes[1] that it became possible to solve even the simplest nonsimilar normal mode system. Many perturbation methods have been developed since to approximate the deviations of nonlinear modes from their corresponding linear modes. In Ref. 2, Caughey et al. identify bifurcation of similar normal modes for a class of strongly nonlinear two-degree-of-freedom systems. Vakakis and Rand use the Mikhlin-Manevich asymptotic approach in Ref. 3 to identify mode localization in systems which exhibit nonsimilar nonlinear normal modes. In Ref. 4 Vakakis and Rand and in Ref. 5 Vakakis make use of Poincaré maps to identify bifurcation and chaos in system which exhibit similar normal modes for separate cases of low and high energy. An asymptotic approach is used by Vakakis and Cetinkaya in Ref. 6 to demonstrate the existance of nonlinear mode localization in perfectly symmetric, weakly coupled structures. King and Vakakis demonstrate an energy-based approach for the determination of nonlinear normal modes of continuous systems in Ref. 7. In Ref. 8, Shaw and Pierre demonstrate an invariant manifold approach to determining nonlinear normal modes of discrete systems. They extend this work in Refs. 9 and 10 to continuous systems, and in Ref. 11 Boivin et al. examine the accuracy and reliability of using the each of the methods described in Refs. 9 and 10.

In Ref. 12 Nayfeh and Nayfeh apply invariant manifold and perturbation methods to a descretized system and the method of multiple scales directly to the partial differential equation of motion. In Ref. 13, they discuss possible errors which may occur when applying discrete analysis to continuous systems. However, only Shaw and Pierre[8] demonstrate the capability of performing coordinate transformation between the physical coordinate system and the nonlinear modal coordinate system. Although algebraically tedious, this method lends itself to programming using algebraic manipulation packages such as Mathematica,®[14] MAPLE,® and MACSYMA.®

The following work applies the method of Shaw and Pierre[8] to find the modal forces acting on nonlinear modal equations for weakly nonlinear systems. It is then shown that the modal forces can be designed to be functions of the modal coordinates, thus providing nonlinear modal feedback control. Breaking the control design problem into a nonlinear modal control problem is analygous to reducing the problem into a number of small single-degree-of-freedom nonlinear control design problems, much in the way that pole placement control design for linear systems reduces control design to the selection of eigenvalues. Thus, the control designer can reduce a large nonlinear control design problem including quadratic, cubic, and higher order nonlinearities into a large number of single-degree-of-freedom design problems. This method also makes control design for known disturbances such as band limited excitation much more intuitive than Lyapunov design since the control designer knows that most of his/her focus should be on controlling nonlinear normal modes whose primary natural frequencies lie in the frequency range of the disturbance. Modal control design can then be accomplished using any one of a number of nonlinear control design techniques applied to the nonlinear modal equations.

II. Overview of Nonlinear Normal Modes

Most dynamicists are, in general, comfortable with the notion of a linear normal mode. The response of a system which is moving in a linear normal mode can be written in the following vector form:

$$\boldsymbol{x}(t) = \boldsymbol{x}|_{t0} e^{-t/\tau} \sin(\omega t + \Phi) \tag{1}$$

where $\boldsymbol{x}|_{t0}$ is the mode shape or the spatial part of the modal dynamics, t is time, τ is the time constant at the mode (infinity for the undamped system), ω is the natural frequency of the mode, and Φ is a phase angle.

Unlike the linear normal mode, nonlinear normal modes are not quite as readily accepted. Nonlinear normal modes are defined to be either similar or nonsimilar. A similar nonlinear normal mode is one in which the mode shape is not dependent on the modal amplitude and, thus, is similar to a linear mode. In a nonsimilar nonlinear normal mode, the mode shape is not linear. For instance, a two-degree-of-freedom system could have the mode shapes $x_2 = f_1(x_1, \dot{x}_1)$ and $\dot{x}_2 = f_2(x_1, \dot{x}_1)$ where f_1 and f_2 are nonlinear functions.

When a system is moving in a mode it moves along a modal line. For a modally damped system, the distance traveled along the modal line decreases with each cycle, and the modal line can change with the amplitude of oscillation. However, since the total energy remains constant for a conservative system, the modal amplitude remains constant, and the modal line will remain constant for all time. A nonmodally damped system moving in a single mode will not follow the same line in both directions but will, instead, follow a curved trajectory that spirals toward static equilibrium. This curved trajectory envelops what would be the undamped modal line.

Received Nov. 24, 1993; revision received July 18, 1994; accepted for publication Aug. 11, 1994.

*Assistant Professor, Department of Mechanical and Materials Engineering. Member AIAA.

†Samuel Herrick Chair Professor, Department of Engineering Science and Mechanics. Associate Fellow AIAA.

Previous work fails to properly define what a nonlinear normal mode is in the case where damping is present. The existing definition[15] for a nonlinear normal mode states that a discrete N-degree-of-freedom system is oscillating in a normal mode if all of the motions are periodic of the same period, all of the coordinates reach their extreme values at the same time, and, for any given amplitude of oscillation, the coordinate displacements can be related by a functional relationship of the form

$$x_i = f_i[u(t)] \tag{2}$$

where x_i is the displacement of the ith coordinate, u is the modal displacement, and f_i is the relationship between them. The definition is too restrictive in the sense that the motions of a system freely oscillating in a modally damped mode will not, in general, be periodic and of the same period. Also, the function f_i must change with the modal amplitude. Shaw and Pierre[8] demonstrate that the modal line is dependent on the modal velocity as well as the modal displacement. Using conservation of energy (for a conservative system, of course) and knowledge of the modal amplitude, the modal velocity contribution may be eliminated. However, Eq. (2) must still be dependent on some quantity other than the modal displacement and, thus, the existing definition is lacking. A simple example of a system oscillating in a modally damped motion is a single-degree-of-freedom mass spring system with velocity squared damping. Each cycle of the motion will exhibit a slightly different period during the decay. Also, complex modes of linear systems are not modes by this definition of nonlinear normal modes since the motion of the system is not in unison. This is accounted for in Ref. 8 by Shaw and Pierre who developed the capability of determining the nonlinear modal equivalent to the linear complex mode. They redefine a normal mode as "a motion which takes place on a two-dimensional invariant manifold in the system's phase space. This manifold has the following properties: it passes through a stable equilibrium point of the system and, at that point, it is tangent to a plane which is an eigenspace of the system linearized about that equilibrium." To provide consistency with linear terminology and improve the definitions of nonlinear modes, the following definitions are proposed.

Definition 1: A system is oscillating in a normal mode if the motion of any point $(u_{\hat{x}}, u_{\hat{y}}, u_{\hat{z}})$ in three-dimensional space $(\hat{x}, \hat{y}, \hat{z})$ can be described by the equation

$$\begin{bmatrix} u_{\hat{x}}(\hat{x}, \hat{y}, \hat{z}) \\ u_{\hat{y}}(\hat{x}, \hat{y}, \hat{z}) \\ u_{\hat{z}}(\hat{x}, \hat{y}, \hat{z}) \end{bmatrix} = \begin{bmatrix} f_{\hat{x}}(u(t), v(t)) \\ f_{\hat{y}}(u(t), v(t)) \\ f_{\hat{z}}(u(t), v(t)) \end{bmatrix} \tag{3}$$

where $(\hat{x}, \hat{y}, \hat{z})$ represents the location of the point on the structure in three-dimensional space; $u_{\hat{x}}, u_{\hat{y}}, and u_{\hat{z}}$ represent the deflections in the x, y, and z directions; $f_{\hat{x}}$, $f_{\hat{y}}$, and $f_{\hat{z}}$ relate the deflections to the modal coordinates u and $v = \dot{u}$. This represents the two-dimensional invariant manifold described by Shaw and Pierre.[8]

Definition 2: If the functions $f_{\hat{i}}$ relating the displacements $u_{\hat{i}}$ to the modal coordinates u and v are linear, and the modal equations in u and v are linear, then the mode is a linear normal mode.

Definition 3: If the functions $f_{\hat{i}}$ relating the displacements $u_{\hat{i}}$ to the modal coordinates u and v are linear, the dependence on v can be eliminated, and the modal equations in u and v are linear, then the mode is a linear equal phase normal mode. These modes are commonly called complex linear modes.

Definition 4: If the functions $f_{\hat{i}}$ relating the displacements $u_{\hat{i}}$ to the modal coordinates are linear and the modal equations in u and v are nonlinear, then the mode is a similar nonlinear normal mode. This corresponds to the definition[1] put forth by Rosenberg that if the modal lines corresponding to a nonlinear normal mode are straight, then the mode is called similar.

Definition 5: If the functions $f_{\hat{i}}$ relating the displacements $u_{\hat{i}}$ to the modal coordinates are nonlinear, then the mode is a nonsimilar nonlinear normal mode. This corresponds to the definition[1] put forth by Rosenberg that if the modal lines corresponding to a nonlinear normal mode are not straight, then the mode is called nonsimilar.

Definition 6: If the trajectory of a nonlinear normal mode passes through static equilibrium, then it is an equal phase nonlinear normal mode. For a linear normal mode, this type of mode is usually called a real mode.

Definition 7: If the trajectory of a nonlinear normal mode does not pass through static equilibrium, then it is a nonequal phase nonlinear normal mode. In a linear system, this type of mode is called a complex mode.

The following two sections describe the method developed by Shaw and Pierre[8] for determining nonlinear normal modes. As will become evident, the method allows for the solution of nonlinear normal modes that were previously undefined by the Rosenberg definition.[1]

III. Normal Modes of Nonlinear Systems

The method used here for determining the nonlinear normal modes of the system is the method of Shaw and Pierre.[8] Readers are referred to this work for determining nonlinear normal modes.

Using the same notation as Shaw and Pierre,[8] the displacements and velocities of a system, $z = [x_1, \dot{x}_1, x_2, \dot{x}_2, \ldots, x_N, \dot{x}_N]^T = [x_1, y_1, x_2, y_2, \ldots, x_N, y_N]^T$, moving in a single nonlinear normal mode can be written as a function of the modal displacment u and the modal velocity v as

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_N \\ y_N \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ a_{12} & a_{22} \\ b_{12} & b_{22} \\ \vdots & \vdots \\ a_{1N} & a_{2N} \\ b_{1N} & b_{2N} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{32}u + a_{42}v & a_{52}v \\ b_{32}u + b_{42}v & b_{52}v \\ \vdots & \vdots \\ a_{3N}u + a_{4N}v & a_{5N}v \\ b_{3N}u + b_{4N}v & b_{5N}v \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{62}u^2 + a_{82}v^2 & a_{72}u^2 + a_{92}v^2 \\ b_{62}u^2 + b_{82}v^2 & b_{72}u^2 + b_{92}v^2 \\ \vdots & \vdots \\ a_{6N}u^2 + a_{82}v^2 & a_{7N}u^2 + a_{9N}v^2 \\ b_{6N}u^2 + b_{8N}v^2 & b_{7N}u^2 + b_{9N}v^2 \end{bmatrix} + \cdots \right\} \begin{bmatrix} u \\ v \end{bmatrix} \tag{4}$$

or more compactly as

$$z = \boldsymbol{m} \begin{bmatrix} u \\ v \end{bmatrix} = [\boldsymbol{m}_0 + \boldsymbol{m}_1(u, v) + \boldsymbol{m}_2(u, v) + \cdots] \begin{bmatrix} u \\ v \end{bmatrix} \tag{5}$$

where $\boldsymbol{m}$ is a $2N \times 2$ matrix, and $\boldsymbol{m}_0$, $\boldsymbol{m}_1$, and $\boldsymbol{m}_2$ are also $2N \times 2$ matrices. The matrix $\boldsymbol{m}_0$ is the linear component of the nonlinear modes, and $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ represent the quadratic and cubic terms, respectively. This representation is not unique but is used to facilitate coordinate transformations later.

The matrix $\boldsymbol{m}_0$ represents the mode shapes common to linear systems. For a linear system, all other matrices $\boldsymbol{m}_j$ are zero. For the linear undamped or modally damped system, the cross terms a_{2i} and b_{1i} are zero whereas $a_{1i} = b_{2i}$ is the usual amplitude ratio relating the ith degree of freedom to the modal amplitude. For a normally damped system, the cross terms are generally nonzero and, thus, represent the effect of complex modes in terms of real numbers.

The complete nonlinear modal matrix M can now be assembled from the modal vectors $\boldsymbol{m}$. The modal matrix $M(\boldsymbol{w})$ is then

$$M(\boldsymbol{w}) = [{}_1\boldsymbol{m} \quad {}_2\boldsymbol{m} \quad {}_3\boldsymbol{m} \quad \cdots \quad {}_N\boldsymbol{m}] \tag{6}$$

where ${}_i\boldsymbol{m}$ represents the modal vector for the ith mode and $\boldsymbol{w} = [u_1, v_1, u_2, v_2, \ldots, u_N, v_N]^T$ where u_i and v_i are the modal displacement and velocity, respectively, for the ith mode. The complete transformation from modal to physical coordinates can now be written

$$z = M(\boldsymbol{w})\boldsymbol{w} = \tilde{\boldsymbol{M}}(\boldsymbol{w}) \tag{7}$$

The matrix $M(w)$ can be subdivided in the same manner as the vector $m(w)$ is in Eqs. (4) and (5). This gives

$$M(w) = M_0 + M_1(w) + M_2(w) + \cdots \tag{8}$$

where M_0 represents the linear terms of $M(w)$, $M_1(w)$ represents the quadratic terms of $M(w)$, and so on.

The coefficients a_{ij} and b_{ij} are different for each mode, in general. It is suggested that the notation a_{ijk} and b_{ijk} be used where the index k represents the mode number.[8] The importance of this form will become apparent in the next section.

IV. Transformation from Physical to Nonlinear Modal Coordinates

It must be made clear that the transformations from physical to nonlinear modal coordinates are an extension of linear theory, but superposition does not apply in the linear sense. The nonlinear coordinate transformation is similar to the well-known linear transformations only in that it allows a multiple-degree-of-freedom system to be viewed as a number of simpler single-degree-of-freedom systems. These single-degree-of-freedom systems can be solved and their solutions combined via the nonlinear coordinate transformations to yield the total system dynamics.

For linear systems, the transformation between modal and physical coordinates is given by the matrix M_0. The transformation from the physical to the modal coordinates for the nonlinear system is not the same as the transformation from the modal to the physical coordinates due to the modal amplitude dependence of the transformation. For the transformation from modal to physical coordinates, it is assumed that the modal amplitudes are known. Since the transformation matrix is written in terms of the modal amplitudes, the transformation can be carried out [see Eq. (7)]. In making the transformation from physical to modal coordinates we cannot simply invert the matrix $M(w)$ to accomplish the transformation because we do not know the modal amplitudes and, therefore, cannot evaluate $M(w)$. The transformation method developed by Shaw and Pierre[8] for systems with only cubic nonlinearities is shown next (a more general method is provided in the Appendix of Ref. 8).

We begin by premultiplying Eq. (6) by $M^{-1}(w)$ and expanding the inverse to yield

$$\begin{aligned} w &= \{M_0 + M_2(w)\}^{-1}z \\ &= \left\{I + M_0^{-1}M_2(w)\right\}^{-1}M_0^{-1}z \\ &= \left\{I - M_0^{-1}M_2(w)\right\}M_0^{-1}z + \left[I + M_0^{-1}M_2(w)\right]^{-1} \\ &\quad \times \left[M_0^{-1}M_2(w)\right]^2 M_0^{-1}z \end{aligned} \tag{9}$$

[Note that since we are considering only the case of cubic nonlinearity, $M_1(w)$ is a zero matrix and can be dropped.] Next, the fourth-order term (the second term) is dropped for the sake of simplicity.

$$w = \left\{I - M_0^{-1}M_2(w)\right\}M_0^{-1}z \tag{10}$$

The right-hand side still has a dependency on w. This can be remedied first by substituting Eq. (9) for w into itself, i.e.,

$$w = \left\{I - M_0^{-1}M_2\left(\left\{I - M_0^{-1}M_2(w)\right\}M_0^{-1}z\right)\right\}M_0^{-1}z \tag{11}$$

Then, since M_2 is quadratic in its argument, the leading order argument is $M_0^{-1}z$ and the dependence on w is pushed on to higher order terms. This results in the transformation

$$w = \left\{I - M_0^{-1}M_2\left(M_0^{-1}z\right)\right\}M_0^{-1}z = N(z)z = \tilde{N}(z) \tag{12}$$

which is correct up to cubic terms in z.

The matrix $N(z)$ now represents the transformation from physical to modal coordinates. Like the coordinate transformation matrix $M(w)$, $N(z)$ can be broken into terms of different order, i.e.,

$$N(z) = N_0 + N_1(z) + N_2(z) + \cdots \tag{13}$$

where N_0 represents the linear part of the transformation, $N_1(z)$ the quadratic part of the transformation, and so on.

V. Transformation of Nonlinear State Equations from Physical to Modal Coordinates

Motivated by the use of the state-space representation in control theory, the nonlinear state-space equations can be written in physical coordinates as

$$\dot{z} = \tilde{A}(z, u) \tag{14}$$

$$y = \tilde{C}(z, \dot{z}) \tag{15}$$

where $\tilde{A}(z)$ is the state function vector, $\tilde{C}(z, \dot{z})$ is the output function vector, and u represents a vector of arbitrary excitations. Taking the derivative of Eq. (7) with respect to time and applying the chain rule yields

$$\dot{z} = \frac{\partial \tilde{M}(w)}{\partial w}\dot{w} \tag{16}$$

Substituting Eqs. (7) and (16) into Eqs. (14) and (15) gives

$$\begin{aligned} \dot{w} &= \left(\frac{\partial \tilde{M}(w)}{\partial w}\right)^{-1}\tilde{A}[\tilde{M}(w), u] \\ &= \tilde{A}_m(w) + u_m(w, u) \end{aligned} \tag{17}$$

and

$$y = \tilde{C}\left[\tilde{M}(w), \left(\frac{\partial \tilde{M}(w)}{\partial w}\right)\dot{w}\right] = \tilde{C}_m(w, \dot{w}) \tag{18}$$

where $u_m(w, u)$ represents the amplitude dependent modal forces, $\tilde{A}_m(w)$ represents the modal functions, and $\tilde{C}_m(w)$ represents the observation function in modal coordinates. Note that the application of Eq. (16) to Eqs. (14) and (15) involves the inverse of the Jacobian of the modal matrix $\tilde{M}(w)$. This inverse will fail if the determinant of the Jacobian of the modal matrix $\tilde{M}(w)$ is not zero. By setting the determinant equal to zero, criterion for failure of the coordinate transformation can be determined in terms of the modal amplitudes.

VI. Transformation of the Nonlinear State Equations from Modal to Physical Coordinates

In transforming the equations from modal coordinates to physical coordinates, the same procedure is used as in the previous section. The equations of motion in modal coordinates are given by

$$\dot{w} = \tilde{A}_m(w) + u_m(w) \tag{19}$$

$$y = \tilde{C}_m(w, \dot{w}) \tag{20}$$

where $u_m(w)$ represents a vector of arbitrary modal forces. From Eq. (12)

$$w = \tilde{N}(z) \tag{21}$$

Taking the time derivative and applying the chain rule to Eq. (21) gives

$$\dot{w} = \frac{\partial \tilde{N}(z)}{\partial z}\dot{z} \tag{22}$$

where $\partial \tilde{N}(z)/\partial z$ is the Jacobian of $\tilde{N}(z)$ with respect to z. Substituting Eqs. (12) and (22) into the modal equations of motion and premultiplying by $(\partial \tilde{N}(z)/\partial z)^{-1}$ yields the state equations in physical coordinates as

$$\begin{aligned} \dot{z} &= \left(\frac{\partial \tilde{N}(z)}{\partial z}\right)^{-1}\tilde{A}_m[\tilde{N}(z)] + \left(\frac{\partial \tilde{N}(z)}{\partial z}\right)^{-1}u_m[\tilde{N}(z)] \\ &= \tilde{A}'(z, u) \end{aligned} \tag{23}$$

and

$$y = \tilde{C}_m\left[\tilde{N}(z), \left(\frac{\partial \tilde{N}(z)}{\partial z}\right)\dot{z}\right] = \tilde{C}'(z, \dot{z}) \tag{24}$$

Here the prime symbol represents the possibility that the transformation does not allow the true equations of motion to be recovered in transforming them from modal to physical coordinates. For example, if a power series approximation for the modes is used, as suggested by Shaw and Pierre,[8] then for practical purposes the series must be truncated for most problems. Since the modes are an approximation of the true modes, only an approximation of the true modal equations will be found. Here we have not distinguished between the true modal equations and the approximations to them since in this work they will always be derived from the equations of motion using a power series and, thus, will almost always be approximate (the exception being similar normal modal systems). Since the equations of motion are derived in physical space and they can also be found again by the transformation from modal coordinates to physical coordinates, it is important to point out that the physical space equations of motion derived from the modal equations will not be as correct as when derived from first principles. However, the backwards transformation (from modal to physical coordinates) can be useful in determining the accuracy of the forwards transformations.

VII. Nonlinear Modal Control

The previously discussed transformations developed for analyzing the forced response of a weakly nonlinear multiple-degree-of-freedom system can now be used to set up a new nonlinear control technique that we will refer to as nonlinear modal control. The approach is to transform the equations of motion from physical space into modal space or decoupled form. Once decoupled, the feedback control law can be designed one mode at a time, subject to implementation constraints, much as suggested by Takahashi et al.[16] and Inman[17] except that both the modal dynamics and the control law are nonlinear. The control law designed in the decoupled or modal coordinate space allows the behavior of each decoupled equation to be tailored to produce a desirable response. However, to implement the control, the form of the control law must also be known in the physical coordinate system. The form of the control law in physical coordinates is transformed into the decoupled or modal coordinate system yielding constraints that must be met in the modal space. Once these constraints are met, the control law can then be transformed into the physical coordinate system as described in the previous section. This procedure is illustrated in the example that follows. Additional examples can be found in Ref. 18.

VIII. Example

Consider the two-degree-of-freedom system of Fig. 1. Also consider the case where only the second degree of freedom can be actuated. The equations of motion for the system are then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -2x_1 - 0.5x_1^3 + x_2 \\ y_2 \\ x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \hat{u}_z$$

and

$$\hat{y} = Iz$$

Using the method of Shaw and Pierre,[8] the equations of motion in modal coordinates are

$$\begin{bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -u_1 - \frac{1}{3}u_1^3 + \frac{1}{4}u_1v_1^2 - \frac{3}{4}u_1^2u_2 - \frac{3}{4}u_1u_2^2 \\ v_2 \\ -3u_2 - \frac{4}{13}u_2^3 + \frac{3}{52}u_2v_2^2 - \frac{3}{4}u_1^2u_2 - \frac{3}{4}u_1u_2^2 \end{bmatrix} + \begin{bmatrix} \hat{u}_{m1} \\ \hat{u}_{m2} \\ \hat{u}_{m3} \\ \hat{u}_{m4} \end{bmatrix}$$

where the modal control forces are

$$\hat{u}_{m1} = -\hat{u}_{m3} = \frac{(-0.5u_1v_1 + 0.115385u_2v_2)\hat{u}_z}{4 + u_1^2 - 1.61538u_2^2 + 2v_1^2 - 0.461538v_2^2} \qquad (25)$$

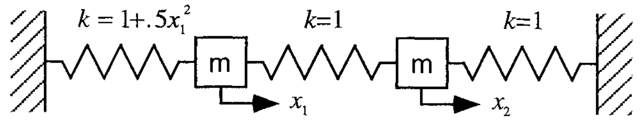

Fig. 1 Two-degree-of-freedom oscillator with a cubic stiffness.

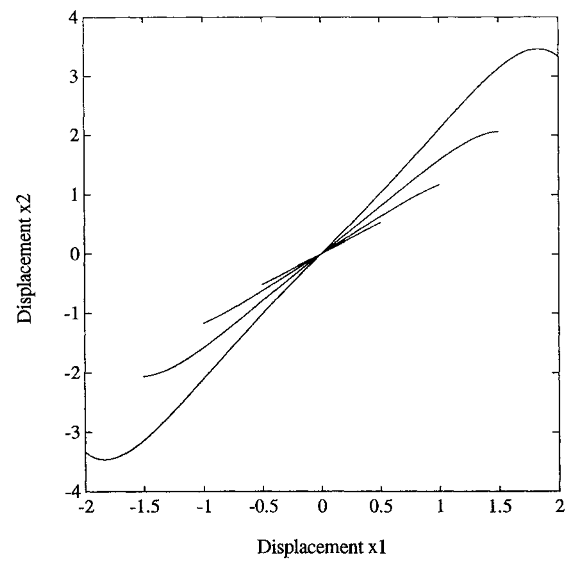

Fig. 2 Mode shape plots of mode 1 for various maximum modal amplitudes (0.2, 0.5, 1.0, 1.5, 2.0).

and

$$\hat{u}_{m2} = -\hat{u}_{m4} = \frac{(2 + 0.5u_1^2 - 0.576923u_2^2 + 0.25v_1^2 - 0.0576923)\hat{u}_z}{4 + u_1^2 - 1.61538u_2^2 + 2v_1^2 - 0.461538v_2^2} \qquad (26)$$

The output equation is given by

$$\hat{y} = \tilde{M}(w) = \begin{bmatrix} u_1 + u_2 \\ v_1 + v_2 \\ u_1 - u_2 + 0.1667u_1^3 + 1.923u_2^3 + 0.25u_1v_1^2 + 0.05769u_2v_2^2 \\ v_1 - v_2 + 0.25v_1^3 + 0.5769v_2^3 + 0.2308u_2^2v_2 \end{bmatrix}$$

and the mode shapes are given by the following.

Mode 1

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \tilde{M}(w)\big|_{u_2=0,v_2=0} = \begin{bmatrix} u_1 \\ v_1 \\ u_1 + \frac{1}{6}u_1^3 + 0.25u_1v_1^2 \\ v_1 + 0.25v_1^3 \end{bmatrix}$$

Mode 2

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \tilde{M}(w)\big|_{u_1=0,v_1=0} = \begin{bmatrix} u_2 \\ v_2 \\ -u_2 + 0.1923u_2^3 + 0.05769u_2v_2^2 \\ -v_2 + 0.05769v_2^3 + 0.2308u_2^2v_2 \end{bmatrix}$$

Figure 2 shows mode shapes (displacement one vs displacement two) for various modal amplitudes of mode 1 (0.2, 0.5, 1.0, 1.5, 2.0). The motions were found by numerically integrating the modal equation of motion and applying the mode shape relation. Note that even the linearization of these modal motions are different depending on the modal amplitude. Figure 3 shows the mode shapes of mode 2 for various modal amplitudes.

The system is ideally observable[18] since

$$w = \tilde{N}(\hat{y})$$

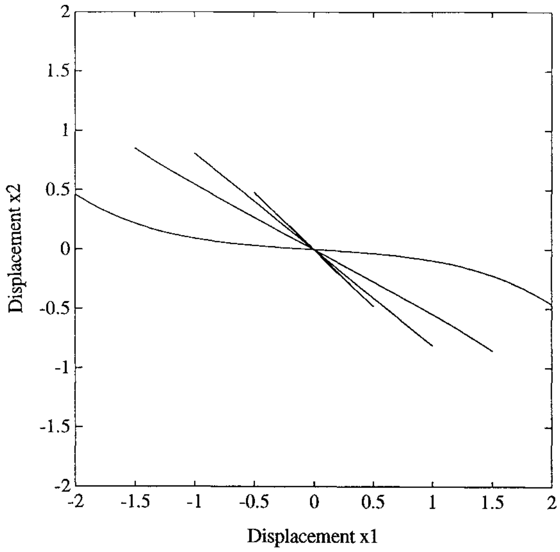

Fig. 3 Mode shape plots of mode 2 for various maximum modal amplitudes (0.2, 0.5, 1.0, 1.5, 2.0).

Also, as long as the modal control law satisfies Eqs. (25) and (26) it will be physically realizable. The relationships between the modal control forces can be written as

$$\hat{u}_{m1} = f\hat{u}_{m2} = -\hat{u}_{m3} = -f\hat{u}_{m4}$$

where

$$f = \frac{-0.5u_1v_1 + 0.115385u_2v_2}{2 + 0.5u_1^2 - 0.576923u_2^2 + 0.25v_1^2 - 0.0576923v_2^2}$$

Since the purpose of this example is to show how a desired nonlinear modal control can be transformed to physical coordinates and applied to the system, a control law will be proposed using heuristic observations. Consider a system in which very small oscillations are considered unimportant but larger oscillations are considered to be detrimental. A damping term could then be introduced which has a small coefficient for small amplitudes and a large coefficient for larger amplitudes. Possible modal control laws could then be

$$\hat{u}_{m2} = -0.1\dot{u}_1\left(1 + \dot{u}_1^2 + u_1^2\right)$$

and

$$\hat{u}_{m4} = -0.1\dot{u}_2\left(1 + \dot{u}_2^2 + 3u_2^2\right)$$

However, because of the constraints of Eqs. (25) and (26), the following alternative is proposed:

$$\hat{u}_{m2} = -0.1\dot{u}_1\left(1 + \dot{u}_1^2 + u_1^2\right) + 0.1\dot{u}_2\left(1 + \dot{u}_2^2 + 3u_2^2\right)$$

and

$$\hat{u}_{m4} = -\hat{u}_{m2}$$

Thus, in addition to the desired control, the negative of the control law intended for mode 1 is applied to mode 2 and the negative of the control law intended for mode 2 is applied to mode 1. It is expected that since these cross terms are small, they should have little impact on the modal equations. Nayfeh and Mook[19] show that for systems having cubic nonlinearities, the effects of the cross terms are negligible when no internal resonance exists.

The control law in modal coordinates is then

$$\begin{bmatrix} 0 \\ \hat{u}_{m2} \\ 0 \\ \hat{u}_{m4} \end{bmatrix}$$

Using Eq. (23) to transform the control law to physical coordinates in terms of the output $\hat{\boldsymbol{y}}$ yields

$$B\hat{u}_z = \begin{bmatrix} 0 \\ \hdashline 0 \\ \hdashline 0.0192308\hat{y}_4(\hat{y}_1\hat{y}_2 + \hat{y}_3\hat{y}_4) \\ +0.030769\hat{y}_4(\hat{y}_3\hat{y}_2 + \hat{y}_1\hat{y}_4) \\ \hdashline -0.2\hat{y}_2 - 0.01\hat{y}_4\left(\hat{y}_1^2 + \hat{y}_3^2\right) + 0.0442308\hat{y}_4\left(\hat{y}_1^2 + \hat{y}_3^2\right) \\ -0.188462\hat{y}_1\hat{y}_3\hat{y}_2 + 0.1\hat{y}_1\hat{y}_3\hat{y}_4 - 0.15\hat{y}_2^2\hat{y}_4 \\ +0.0230769\hat{y}_2\hat{y}_4^2 - 0.0403846\hat{y}_4^3 \end{bmatrix}$$

From the equations of motion, $B\hat{u}_z$ clearly cannot have this form. All but the last element of $B\hat{u}_z$ must be identically zero. The error is believed to be from the approximate inverse (from modal to physical) coordinate transformation ($\tilde{N}(z)$) as discussed in Sec. VI. Since the fourth element of $B\hat{u}_z$ should be equal to $\hat{u}_z$, $\hat{u}_z$, is taken to be

$$\hat{u}_z = -0.2\hat{y}_4 - 0.1\hat{y}_4\left(\hat{y}_1^2 + \hat{y}_3^2\right) + 0.0442\hat{y}_2\left(\hat{y}_1^2 + \hat{y}_3^2\right)$$
$$- 0.1885\hat{y}_1\hat{y}_3\hat{y}_2 + 0.1\hat{y}_1\hat{y}_3\hat{y}_4 - 0.15\hat{y}_2^2\hat{y}_4 + 0.0231\hat{y}_2\hat{y}_4^2$$
$$- 0.0404\hat{y}_4^3 - 0.00769\hat{y}_2^3$$

For simple comparison, an approximate linear modal control has been derived using a linearized set of equations. The linear control derived is identical to the linear part of the nonlinear control law. Note that neither the linear nor the nonlinear control has been optimized in any way and, thus, the comparison demonstrates only the increased flexibility of the nonlinear modal control method as compared to the corresponding linear modal control. Simulations of the controlled steady-state response and transient response are shown in the following plots. The displacement plotted is that of x_1. Figures 4 and 5 show the transient response from an initial condition of $x_1 = 1$ and all other states equal to zero. Clearly the control forces are greater at higher amplitudes for the nonlinear control than for the linear control. Also, the motion of the first mass is reduced much more quickly using the nonlinear modal control than the linear control. Figures 6 and 7 show the steady state response to a 1-rad/s sinusoidal force acting on mass 2 with an amplitude of 0.01 (low amplitude) and zero initial conditions. The concurrence of the trajectories demonstrates that the goal of providing only a linear energy dissipation at low amplitudes has been met. Figures 8 and 9 show the steady-state response to a 1-rad/s sinusoidal force on mass 2 with an amplitude of 1 (high amplitude). Here the effect

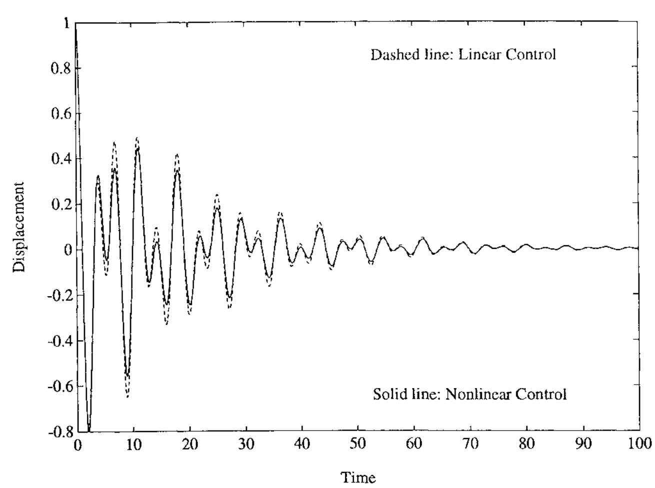

Fig. 4 Transient response, displacement of x_1 vs time.

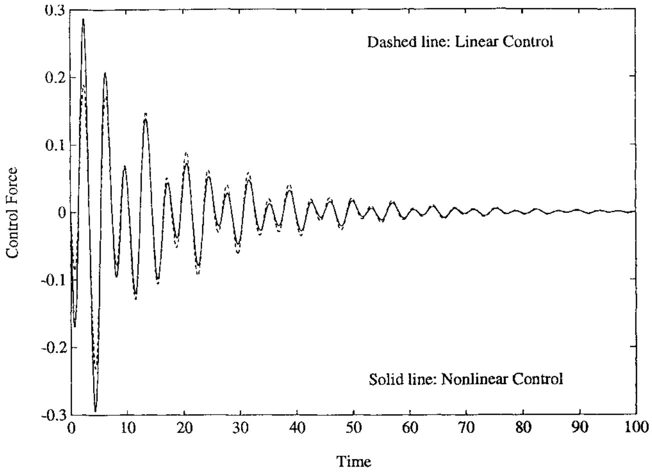

Fig. 5 Transient response, control force vs time.

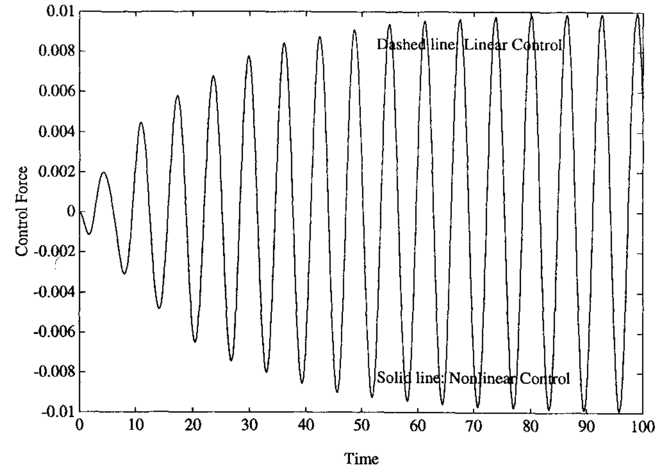

Fig. 7 Steady-state response, control force vs time, 1 rad/s, excitation amplitude = 0.01.

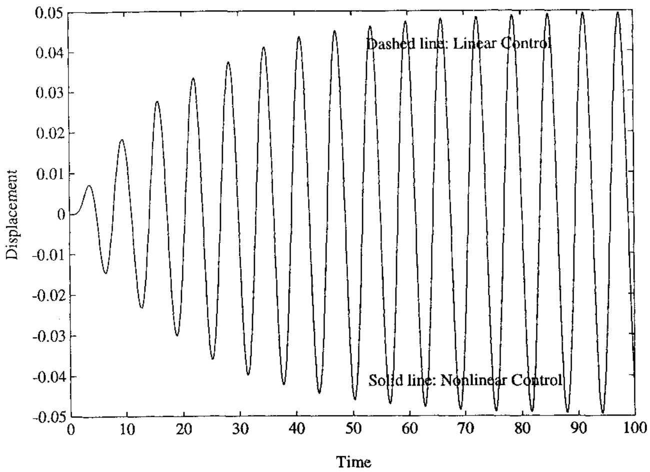

Fig. 6 Steady-state response, displacement of x_1 vs time, 1 rad/s, excitation amplitude = 0.01.

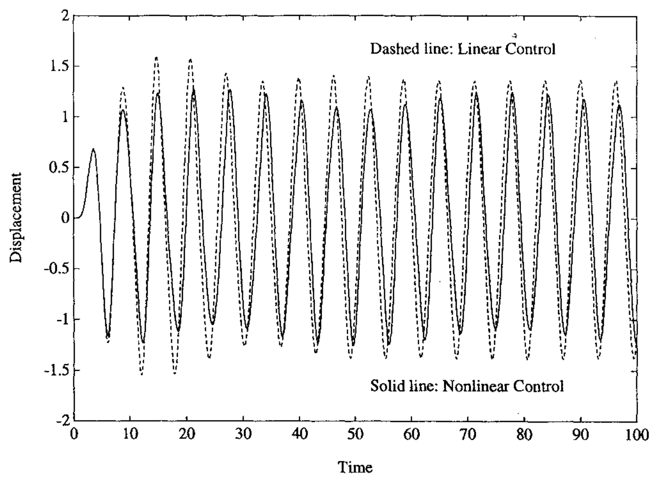

Fig. 8 Steady-state response, displacement of x_1 vs time, 1 rad/s, excitation amplitude = 1.

of the nonlinear control is clear. At higher amplitudes, the nonlinear control provides additional damping, beyond that of the linear control and, therefore, reduces the steady-state response. Thus, this example shows two major points. The first is that depending on system requirements, nonlinear control provides an improved closed-loop performance compared to the linear control. This fact can be observed in most systems when finding the optimal open-loop control. Generally, any reasonably sized (low-order) linear controller does not match the capability of the optimal open-loop control (robustness and stability being neglected). The second is that modal control laws, be they linear or nonlinear, can be successfully derived for nonlinear modal systems even if the modes are nonsimilar.

To determine how modal the nonlinear control law is, the nonlinear normal modes of the controlled system were determined using the invariant manifold method of Shaw and Pierre (i.e., all control forces were included as inherent system forces in performing the nonlinear modal analysis). The results are as follows.

Using the method of Shaw and Pierre,[8] the equations of motion in modal coordinates to cubic order are

$$\begin{bmatrix} \dot{u}_1 \\ \dot{v}_1 \\ \dot{u}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_1 - 0.0758u_1^2u_2 - 0.0758u_1u_2^2 \\ \hdashline -1.005u_1 - 0.1005v_1 - 0.321u_1^3 - 0.382u_1^2v_1 + 0.241u_1v_1^2 \\ -0.761u_1^2u_2 - 0.762u_1u_2^2 - 0.00460u_1u_2v_1 - 0.211u_2^2v_1 \\ -0.0352u_2v_1^2 - 0.0994v_1^3 + 0.00567u_1u_2v_2 + 0.00702u_1v_1v_2 \\ +0.0181u_2v_1v_2 - 0.0407v_1^2v_2 - 0.0157u_1v_2^2 - 0.00914v_1v_2^2 \\ \hdashline v_2 + 0.0758u_1^2u_2 + 0.0758u_1u_2^2 \\ \hdashline -2.985u_2 - 0.0995v_2 - 0.343u_2^3 - 0.739u_1^2u_2 - 0.738u_1u_2^2 \\ +0.00460u_1u_2v_1 + 0.0211u_2^2v_1 + 0.0352u_2v_1^2 \\ -0.00567u_1u_2v_2 - 0.287u_2^2v_2 - 0.00702u_1v_1v_2 - 0.0181u_2v_1v_2 \\ +0.0407v_1^2v_2 + 0.0157u_1v_2^2 + 0.0370u_2v_2^2 + 0.00914v_1v_2^2 \\ -0.0999v_2^3 \end{bmatrix}$$

The mode shapes are given by the following:
Mode 1:

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \tilde{M}(w)\big|_{u_2=0,v_2=0} = \begin{bmatrix} u_1 \\ v_2 \\ 0.995u_1 - 0.1005v_1 + 0.179u_1^3 - 0.382u_1^2 v_1 + 0.241u_1 v_1^2 - 0.0994v_1^3 \\ 0.1010u_1 + 1.005v_1 + 0.0706u_1^3 + 0.0601u_1^2 v_1 + 0.1505u_1 v_1^2 + 0.281v_1^3 \end{bmatrix}$$

Mode 2:

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \tilde{M}(w)\big|_{u_1=0,v_1=0} = \begin{bmatrix} u_2 \\ v_1 \\ -0.985u_1 - 0.0995v_1 + 0.157u_1^3 - 0.287u_1^2 v_1 + 0.0370u_1 v_1^2 - 0.0999v_1^3 \\ 0.297u_1 + 0.975v_1 + 0.892u_1^3 + 0.307u_1^2 v_1 + 0.309u_1 v_1^2 + 0.0768v_1^3 \end{bmatrix}$$

Observing the nonlinear mode shapes of the controlled system, one can see that for each mode the displacement of the second mass is a function of the displacement and the velocity of the first mass. Similarly, the velocity of the second mass is a function of the displacement and the velocity of the first mass. Neither of these modes passes through equilibrium for a nonzero modal amplitude. Thus, the applied control has induced nonequal phase normal modes in the system. Intuitively, from knowledge of linear systems, this makes sense. Since the control law is dissipative in nature and applied to only one mass, and the original system is undamped, the resulting controlled system should exhibit nonequal phase normal modes (a nonproportionally damped linear system will usually exhibit complex modes unless certain conditions on the mass, damping, and stiffness matrices are met).[20]

Observing the nonlinear modal equations of the controlled system and comparing them to the desired modal equations, one can see that the desired form of the modal control law has not been achieved. Although some of the desired damping terms are present, the coefficients of some are lower than desired and some are missing. Other terms that were not desired were also created by applying the control law. However, simulations of the controlled system show that the desired control objectives have been achieved. The reason for this discrepancy is believed to be due to the significant cross coupling designed into the control law. The control law was, in reality, significantly nonmodal to the extent that it also changed the modal coordinate system and, thus, the nonlinear normal modes of the controlled system are different from the nonlinear normal modes of the uncontrolled system. However, since the controlled system could also be represented by the designed nonlinear modally controlled modal equations and the nonlinear normal modes of the uncontrolled system, the modal response characteristics of the controlled system are the same as the designed modal response characteristics.

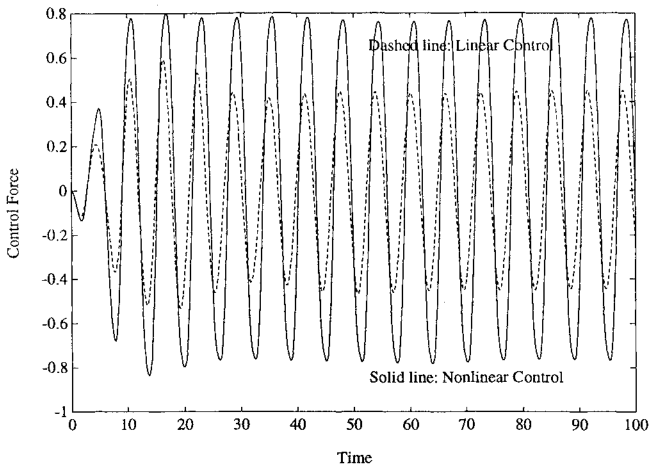

Fig. 9 Steady-state response, control force vs time, 1 rad/s, excitation amplitude = 1.

IX. Conclusion

The invariant manifold method has been extended to the forced response case and now includes the output equation as well as the state equation. The appropriate transformations have been derived and applied to a two-degree-of-freedom nonlinear oscillator illustrating that a successful feedback control law can be designed for a nonlinear system based on nonlinear modal control techniques. In addition, several new definitions have been put forth to allow nonlinear modal nomenclature to more precisely agree with the nonconservative linear nomenclature.

Acknowledgments

This work was supported in part by Army Research Office Grant DAAL0391 G0063 under the supervision of Gary Anderson. The author acknowledges the reviewers for their helpful comments and suggestions.

References

[1]Kuerti, G. (ed.), *Advances in Applied Mechanics*, Academic, New York, Vol. 9, 1966, pp. 155–242.

[2]Caughey, T. K., Vakakis, A. F., and Sivo, J., "Analytical Study of Similar Normal Modes and Their Bifurcations in a Class of Strongly Non-Linear Systems," *International Journal of Non-Linear Mechanics*, Vol. 25, No. 5, 1990, pp. 521–533.

[3]Vakakis, A. F., and Rand, R. H.,"Normal Modes and Global Dynamics of a Two-Degree-Of-Freedom Non-Linear System–Low Energies," *International Journal of Non-Linear Mechanics*, Vol. 27, No. 5, 1992, pp. 861–874.

[4]Vakakis, A. F., and Rand, R. H.,"Normal Modes and Global Dynamics of a Two-Degree-Of-Freedom Non-Linear System–High Energies," *International Journal of Non-Linear Mechanics*, Vol. 27, No. 5, 1992, pp. 875–888.

[5]Vakakis, A. F.,"Non-Similar Normal Oscillations In a Strong Non-Linear Discrete System," *Journal of Sound and Vibration*, Vol. 158, No. 2, 1992, pp. 341–361.

[6]Vakakis, A. F., and Cetinkaya, C., "Mode Localization in a Class of Multidegree-of-Freedom Nonlinear Systems with Cyclic Symmetry," *SIAM Journal of Applied Mathematics*, Vol. 53, Feb. 1993, pp. 265–282.

[7]King, M. E., and Vakakis, A. F., "An Energy-Based Formulation for Computing Nonlinear Normal Modes in Undamped Continuous Systems," *Journal of Vibration and Acoustics*, Vol. 116, No. 3, 1994, pp. 332–340.

[8]Shaw, S. W., and Pierre, C., "Normal Modes for Nonlinear Vibratory Systems," *Journal of Sound and Vibration*, Vol. 164, June 1993, pp. 85–124.

[9]Shaw, S. W., and Pierre, C., "On Nonlinear Normal Modes," *Proceedings of the Winter Annual Meeting of the American Society of Mechanical Engineers*, Nov. 1992, pp. 1–5.

[10]Shaw, S. W., and Pierre, C., "Normal Modes of Vibration for Non-Linear Continuous Systems," *Journal of Sound and Vibration*, Vol. 169, No. 3, 1994, pp. 319–347.

[11]Boivin, N., Pierre, C., and Shaw, S. W., "Non-Linear Normal Modes, Invariance, and Modal Dynamics Approximations of Non-Linear Systems," *Nonlinear Dynamics* (to be published).

[12]Nayfeh, A. H., and Nayfeh, S. A., "On Nonlinear Normal Modes of Continous Systems," *Journal of Vibration and Acoustics*, Vol. 116, No. 1, 1994, pp. 129–136.

[13]Nayfeh, A. H., and Nayfeh, S. A., "Nonlinear Normal Modes of

Continuous Systems with Quadratic Nonlinearities," *Journal of Vibration and Acoustics* (to be published).

[14]Wolfram, S., *Mathematica, a System for Doing Mathematics by Computer*, Addison–Wesley, New York, 1991, Appendix A.

[15]Vakakis, A. F., "Analysis and Identification of Linear and Nonlinear Normal Modes in Vibrating Systems," Ph.D. Thesis, California Inst. of Technology, Pasadena, CA, July 1990, pp. 1–206.

[16]Takahashi, Y., Thal-Larsen, H., Goldberg, E., Loscutoff, W. V., and Ragetly, P. R., "Mode Oriented Design Viewpoint for Linear Lumped-Parameter Multivariable Control Systems," *Journal of Basic Engineering*, Vol. 90, No. 1, 1968, pp. 222–230.

[17]Inman, D. J., *Vibration, With Control, Measurement, and Stability*, Prentice–Hall, New York, 1989, pp. 169–172.

[18]Slater, J. C., "Nonlinear Modal Control," Ph.D. Thesis, State Univ. of New York at Buffalo, Buffalo, NY, June 1993, pp. 1–24.

[19]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations*, Wiley, New York, 1979, pp. 390, 391.

[20]Caughey, T. K., and O'Kelly, M. E. J., "Classical Normal Modes in Damped Linear Systems," *Journal of Applied Mechanics*, Vol. 32, No. 2, pp. 583–588.